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Whittaker–Kotelnikov–Shannon Sampling Theorem and Aliasing Error

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Let $B_{\sigma,p}$, $1 \leqslant p \leqslant \infty$, be the set of all functions from $L_p(\mathbb{R})$ which can be continued to entire functions of exponential type $\leqslant \sigma$. The well known Whittaker–Kotelnikov–Shannon sampling theorem states that every $f \in B_{\sigma,2}$ can be represented as

$$f(x) = \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin \sigma(x - k\pi/\sigma)}{\sigma(x - k\pi/\sigma)}, \qquad \sigma > 0,$$

in norm $L_2(\mathbb{R})$. We prove that it is also true for all $f \in B_{\sigma,p}$, $1 < p < \infty$, in norm $L_p(\mathbb{R})$. From this, we further prove that if $f(x) = O(\Psi(x))$, where $\Psi(x) \in L_p(\mathbb{R})$, $\Psi(x) \geqslant 0$ is even and non-increasing on $[0, \infty)$, and $f(x)$ is Riemann integrable on every finite interval, then the aliasing error of f, i.e., $f(x) - \sum_{k \in \mathbb{Z}} f(k\pi/\sigma)$ $\sin \sigma(x - k\pi/\sigma)[\sigma(x - k\pi/\sigma)]^{-1}$, converges to zero in $L_p(\mathbb{R})$, $1 < p < \infty$, when $\sigma \to +\infty$. If $f \in L_p^r(\mathbb{R})$, $r \in \mathbb{N}$, we also determine the error bound of its aliasing error.

1. INTRODUCTION

Let E be a finite interval or the real axis $\mathbb{R}$ and denote by $L_p(E)$, $1 \leqslant p \leqslant \infty$, the classical Lebesgue space with the usual norm. We say a function f is bandlimited if its Fourier transform has finite support. The well known Whittaker–Kotelnikov–Shannon sampling theorem which plays an important role in communication, information theory, control theory, and data processing [1, 2] states that every signal function which is bandlimited to $[-\sigma, \sigma]$ can be completely reconstructed from its sampled values $f(k\pi/\sigma)$. We formulate it as

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THEOREM A [2]. *Let $f \in L_2(\mathbb{R}) \cap C(\mathbb{R})$ and the support of the Fourier transform $\hat{f}$ of f be contained in $[-\sigma, \sigma]$. Then*

(a) $f(x) = \sum f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma)$, *for all* $x \in \mathbb{R}$,

(b) $\lim_{m \to \infty} \| f(x) - \sum_{|k| \leqslant m} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma) \|_{2(\mathbb{R})} = 0$, *where* $\operatorname{sinc} x = x^{-1} \sin x$ *for* $x \neq 0$, *and* 1 *for* $x = 0$. $\sum_{k \in \mathbb{Z}} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma)$ *is usually named as a Whittaker cardinal series.*

During the past hundred years or so many attempts have been made to generalize Theorem A in a purely mathematical as well as in a practical engineering sense. For example, concerning functions which are not a priori bandlimited, one approximates by bandlimited functions and considers estimates for the error. The papers of Butzer, Higgins, and Splettstösser [1–4] have given an extensive list of references with respect to this direction. In particular, Brown [5] has proved that

THEOREM B [5]. *Let $f \in C(\mathbb{R}) \cap L_p(\mathbb{R})$, $1 \leqslant p \leqslant 2$, $\hat{f} \in L(\mathbb{R})$. Then*

(a) $|\sum_{k \in \mathbb{Z}} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma) - f(x)| \leqslant \sqrt{2/\pi} \int_{|t| \geqslant \sigma} |\hat{f}(t)| \, dt$,

(b) $\lim_{\sigma \to \infty} \sum_{k \in \mathbb{Z}} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma) = f(x)$

uniformly on $\mathbb{R}$, where $\hat{f}(x)$ is the Fourier transform of $f(x)$.

Remark 1. In the language of electrical engineers, the difference $f(x) - \sum_{k \in \mathbb{Z}} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma)$ is called the aliasing error.

DEFINITION 1. Let $g(z)$ be an entire function, $\sigma > 0$; if for every $\varepsilon > 0$, there is a constant $A = A(\varepsilon)$ such that

$$|g(z)| \leqslant A \exp(\sigma + \varepsilon) |z|, \quad \forall z \in \mathbb{C}, \tag{1.1}$$

then $g(z)$ is said to be an entire function of exponential type σ. Denote by E_σ the class of all entire functions of exponential type σ, and let B_σ be the subset of all functions of E_σ which are bounded on $\mathbb{R}$; finally, let

$$B_{\sigma, p} = B_\sigma \cap L_p(\mathbb{R}), \quad 1 \leqslant p \leqslant \infty, \qquad B_{\sigma, \infty} := B_\sigma, \quad \sigma > 0. \tag{1.2}$$

According to Schwartz's theorem [6, p. 110]

$$B_{\sigma, p} = \{ f \in L_p(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\sigma, \sigma] \}, \tag{1.3}$$

the $\hat{f}(x)$ in (1.3) means the Fourier transform of $f(x)$ in the sense of generalized functions [6, p. 30]. For the case $p = 2$, it is the classical Paley–Wiener theorem, therefore, in view of the Schwartz theorem, if a function $f \in L_p(\mathbb{R})$ is bandlimited in $[-\sigma, \sigma]$, then $f \in B_{\sigma, p}$. Rahman and Vértesi [7] have considered the convergence of Lagrange interpolation of some non-periodic function by entire functions of exponential type $\sigma > 0$ in the points $k\pi/\sigma$, $k \in \mathbb{Z}$. In order to relate their results, we need the following definitions:

DEFINITION 2 [7]. Given $1 \leqslant p < \infty$, we denote by $\mathscr{F}_p(\delta)$ the set of all measurable functions $f: \mathbb{R} \to \mathbb{C}$ with

$$f(x) = O((1+|x|)^{-1/p-\delta}), \qquad x \in \mathbb{R}\ (|x| \to \infty) \tag{1.4}$$

for some $\delta > 0$, and by $\mathscr{F}_p$ the union $\bigcup_{\delta>0} \mathscr{F}_p(\delta)$. Clearly $\mathscr{F}_p \subset L_p(\mathbb{R})$.

DEFINITION 3 [7]. We denote by $\mathfrak{R}$ the set of all functions $f: \mathbb{R} \to \mathbb{C}$ which are Riemann integrable on every finite interval.

Rahman and Vértesi [7] have proved

THEOREM C [7]. *Let $f \in \mathscr{F}_p \cap \mathfrak{R}$, $1 < p < \infty$. Then*

$$\left\| f(x) - \sum_{k \in \mathbb{Z}} f\left(\frac{k\pi}{\sigma}\right) \operatorname{sinc} \sigma(x - k\pi/\sigma) \right\|_{p(\mathbb{R})} \to 0, \qquad 1 < p < \infty.$$

Remark 2. (1) The notation T_n denotes the class of all trigonometric polynomials of degree $\leqslant n$. Let $f: \mathbb{R} \to \mathbb{C}$ be a continuous, 2π-periodic function, and denote by $t_n(f, \cdot)$ the trigonometric interpolatory polynomial of degree not exceeding n with $t_n(f; x_{n,k}) = f(x_{n,k})$ in the points $x_{n,k} = 2k\pi/(2n+1)$, $k = 0, \pm 1, ..., \pm n$. It was shown by Marcinkiewicz [8] that

$$\lim_{m \to \infty} \int_0^{2\pi} |f(x) - t_n(f, x)|^p \, dt = 0, \qquad p > 0. \tag{1.5}$$

It is known that $B_n = T_n$ [9, pp. 175–180], hence Marcinkiewicz's result was a motivation for Rahman and Vértesi's paper.

(2) Reference [7] points out that there is a continuous function $f^*: \mathbb{R} \to \mathbb{C}$ which has compact support and

$$\lim_{\sigma \to +\infty} \left\| f^*(x) - \sum_{k \in \mathbb{Z}} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma) \right\|_{\infty(\mathbb{R})} = +\infty.$$

The above results are the motivation for considering the following two problems: First, can be completely reconstruct $f \in B_{\sigma,p}$, $p \in (1, \infty)\backslash 2$, from its sampled values $f(k\pi/\sigma)$ in $L_p(\mathbb{R})$ metric? Second, how large is the aliasing error for differentiable functions which belong to $L_p(\mathbb{R})$? It is the purpose of this paper to consider these two questions. Our main results are the following:

THEOREM 1. *Let* $f \in B_{\sigma,p}$, $1<p<\infty$. *Then*

(a) $f(x)=\sum_{k\in\mathbb{Z}} f(k\pi/\sigma)\operatorname{sinc}\sigma(x-k\pi/\sigma)$, $\forall x\in\mathbb{R}$, *and the series* $\sum_{k\in\mathbb{Z}} f(k\pi/\sigma)\operatorname{sinc}\sigma(x-k\pi/\sigma)$ *converges uniformly on* $\mathbb{R}$.

(b) $\|f(x)-\sum_{|k|\leqslant m} f(k\pi/\sigma)\operatorname{sinc}\sigma(x-k\pi/\sigma)\|_{p(\mathbb{R})}\to 0$, $m\to\infty$,

(c) *there is a constant* C_p *which depends on p only such that*

$$\|f\|_{p(\mathbb{R})}\leqslant C_p\left(\frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p}.$$

Remark 3. (1) The parts (a), (b) of Theorem 1 are generalizations of the Whittaker–Kotelnikov–Shannon sampling theorem (see Theorem A) in $B_{\sigma,p}$, $1<p<\infty$.

(2) Part (c) of Theorem 1 is a generalization of the Marcinkiewicz inequality on $B_{\sigma,p}$, $1<p<\infty$.

(3) If $1\leqslant p<2$, then $B_{\sigma,p}\subset B_{\sigma,2}$ [8, Theorem 8.3.5], therefore, if $1\leqslant p<2$, Part (a) of Theorem 1 is contained in Theorem A.

(4) Rahman and Vértesi [7] have proved that if $f\in B_{\sigma,p}\cap\mathscr{F}_p(\delta)$, $\delta>0$, then Part (c) of Theorem A is valid.

Let l_p, $1\leqslant p\leqslant\infty$, be the Banach space of double infinite bounded sequences with the usual norm

$$\|y\|_{l_p}:=\left(\sum_{j\in\mathbb{Z}}|y_j|^p\right)^{1/p},\qquad 1\leqslant p<\infty,$$
$$\|y\|_{l_\infty}=\sup_{j\in\mathbb{Z}}|y_j|. \tag{1.6}$$

THEOREM 2. (a) *Let* $y=\{y_k\}_{k\in\mathbb{Z}}$, $y\in l_p$, $1<p<\infty$. *Then there is a unique* $g\in B_{\sigma,p}$, *interpolating the given data* $y=\{y_k\}_{k\in\mathbb{Z}}$ *in the points* $k\pi/\sigma$, $k\in\mathbb{Z}$, *and* $g(x)$ *is represented by*

$$g(x)=\sum_{k\in\mathbb{Z}} g(k\pi/\sigma)\operatorname{sinc}\sigma(x-k\pi/\sigma),\qquad \textit{for all}\quad x\in\mathbb{R}, \tag{1.7}$$

and the series $\sum_{k\in\mathbb{Z}} g(k\pi/\sigma)\operatorname{sinc}\sigma(x-k\pi/\sigma)$ *converges uniformly on* $\mathbb{R}$.

(b) *If there is an entire function* $g\in B_{\sigma,p}$, $1\leqslant p\leqslant\infty$, *such that* $g(k\pi/\sigma)=y_k$, $k\in\mathbb{Z}$, *then* $y=\{y_k\}_{k\in\mathbb{Z}}\in l_p$.

Let $f:\mathbb{R}\to\mathbb{C}$ be a measurable function such that $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}}\in l_p$, $1<p<\infty$, then by Theorem 2 there is an interpolation operator $L_\sigma(f,x)\in B_{\sigma,p}$, such that

$$L_\sigma(f,k\pi/\sigma)=f(k\pi/\sigma),\qquad k\in\mathbb{Z}.$$

We write also

$$L_\sigma(f) := L_\sigma(f, \cdot).$$

THEOREM 3. *Let* $f \in L_p(\mathbb{R})$, $1 < p < \infty$, *and* $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}} \in l_p$ *for all* $\sigma > 0$. *Then* $\|f - L_\sigma(f)\|_{p(\mathbb{R})} \to 0$ *if and only if there is a sequence* $\{g_\sigma\} \subset B_{\sigma,p}$ *such that the following two conditions are both satisfied simultaneously.*

(a) $\|f - g_\sigma\|_{p(\mathbb{R})} \to 0$, $\sigma \to +\infty$,

(b) $((\pi/\sigma) \sum_{k\in\mathbb{Z}} |f(k\pi/\sigma) - g_\sigma(k\pi/\sigma)|^p)^{1/p} \to 0$, $\sigma \to +\infty$.

DEFINITION 4. Let $f\colon \mathbb{R} \to \mathbb{C}$ be a measurable function. We say $f \in \Omega_p$, $1 \leqslant p < \infty$, if there is a nonnegative, even, nonincreasing on $[0, \infty)$ function $h \in L_p(\mathbb{R})$, such that

$$|f(x)| = O(h(x)), \qquad \forall x \in \mathbb{R}. \tag{1.8}$$

Remark 4. (1) It is clear that $\Omega_p \subsetneq L_p(\mathbb{R})$ and $\mathscr{F}_p \subsetneq \Omega_p$, for example,

$$f(x) = (2+|x|)^{-1/p} (\log(2+|x|))^{-1/p-\beta} \in \Omega_p, \qquad \beta > 0,$$

and $f \notin \mathscr{F}_p(\delta)$ for any $\delta > 0$.

(2) If $f \in \Omega_p$, then $\{f(k\pi/\sigma)\}_{k\in\mathbb{Z}} \in l_p$, $1 \leqslant p < \infty$, for all $\sigma > 0$.

THEOREM 4. *Let* $f \in \Omega_p \cap \mathfrak{R}$, $1 < p < \infty$. *Then*

$$\|f - L_\sigma(f)\|_{p(\mathbb{R})} \to 0, \qquad \sigma \to +\infty.$$

Remark 5. Theorem 4 extends Rahman and Vértesi's result [7].

Denote by $L_p^r(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, the subspace of functions f in $L_p(\mathbb{R})$ for which the $(r-1)$th derivative of f exists and is locally absolutely continuous on $\mathbb{R}$, and for which $\|f^{(r)}\|_{p(\mathbb{R})}$ is finite; further, let

$$W_p^r(\mathbb{R}) := \{f \in L_p^r(\mathbb{R})\colon \|f^{(r)}\|_{p(\mathbb{R})} \leqslant 1\}.$$

Given $1 \leqslant p \leqslant \infty$, the function

$$\omega(f, t)_{p(\mathbb{R})} = \sup_{|h| \leqslant t} \|g(\cdot + h) - g(\cdot)\|_{p(\mathbb{R})}$$

is called the modulus of smoothness of f in $L_p(\mathbb{R})$. If $f \in L_p(\mathbb{R})$ is a differentiable function, we obtain a bound for the aliasing error of the function f as follows:

THEOREM 5. *Let* $f \in L_p^r(\mathbb{R})$, $r \in \mathbb{N}$, $1 < p < \infty$, $\sigma > 1$. *Then there is a constant* $C_{r,p}$ *which depends on r and p only such that*

$$\|f - L_\sigma(f)\|_{p(\mathbb{R})} \leqslant C_{r,p} \sigma^{-r} \omega\left(f, \frac{1}{\sigma}\right)_{p(\mathbb{R})}.$$

Remark 6. (1) By virtue of [6, p. 168], if $f \in L_p^r(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, then

$$E_\sigma(f)_{p(\mathbb{R})} := \inf_{g \in B_{\sigma,p}} \|f - g\|_{p(\mathbb{R})} \leqslant C_{r,p} \sigma^{-r} \omega\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\mathbb{R})}.$$

(2) In view of [10, 11], the order of the σ-average width in the sense of Kolmogorov and linear width of $W_p^r(\mathbb{R})$, $1 < p < \infty$, is equal to σ^{-r}; therefore, the interpolating operator $L_\sigma(f)$ gives an optimal linear algorithm of these widths.

(3) Ries and Stens [16] and Splettstösser *et al.* [17] (see also [18]) gave the following estimate.

Let f be a locally Riemann integrable function such that $|f(x)| = O(|x|^{-\gamma})$, $|x| \to \infty$, for some $\gamma > 0$. If f is continuous at x_0 and of bounded variation in a neighborhood of x_0, or if f satisfies a Dini–Lipschitz condition in a neighborhood of x_0, i.e.,

$$\lim_{\delta \to 0^+} \omega(f, \delta, C[x_0 - \varepsilon, x_0 + \varepsilon]) \log\left(\frac{1}{\delta}\right) = 0, \tag{1.9}$$

where ω denotes the usual modulus of continuity, then $L_\sigma(f, x_0) \to f(x_0)$. If (1.9) is replaced by $\omega(f, \delta; C(\mathbb{R})) = O(\delta^\alpha)$, $\delta \to 0^+$, for some $\alpha > 0$, then

$$\|f - L_\sigma(f)\|_{C(\mathbb{R})} = O(\sigma^{-\alpha} \log \sigma) \qquad (\sigma \to +\infty)$$

where, as usual, $C(\mathbb{R})$ denotes the set of all real- or complex-valued, uniformly continuous and bounded functions f, defined on $\mathbb{R}$, endowed with the supremum norm $\|f\|_{C(\mathbb{R})}$.

2. SAMPLING THEOREM

In the following, $C_{r,p}$ and C_r stand for two constants which only depend on r and p or r respectively, and they may vary from one equation to the other.

Let $K(x)$ be the unique integer satisfying $K(x)-\frac{1}{2}\leqslant x<K(x)+\frac{1}{2}$, and let

$$Hy(x)=\sum' y_k(x-k)^{-1}, \tag{2.1}$$

where $\sum'$ denotes that the sum is taken over those $k\in\mathbb{Z}$ for which $k\neq K(x)$. $Hy(x)$ is named the mixed Hilbert transform of the sequence $y=\{y_k\}_{k\in\mathbb{Z}}$.

LEMMA 2.1 [12]. *Let* $1<p<\infty$. *Then* $Hy(x)$ *is a linear bounded operator from* $l_p\to L_p(\mathbb{R})$, *i.e.*,

$$\|Hy\|_{p(\mathbb{R})}\leqslant C_p\,\|y\|_{l_p}, \qquad \textit{for all}\quad y\in l_p. \tag{2.2}$$

Let $L_\sigma y:=\sum_{k\in\mathbb{Z}} y_k \operatorname{sinc}\sigma(x-k\pi/\sigma)$, and let

$$\|L_\sigma\|_{p(\mathbb{R})}:=\sup\{\|L_\sigma y(x)\|_{p(\mathbb{R})}: \|y\|_{l_p}\leqslant 1\}. \tag{2.3}$$

$\|L_\sigma\|_{p(\mathbb{R})}$ is called the Lebesgue constant of the Whittaker operator $L_\sigma y(x)$. Following the idea of [12], we have

LEMMA 2.2. *Let* $1<p<\infty$. *Then*

$$\|L_\sigma\|_{p(\mathbb{R})}\leqslant\left(\frac{\pi}{\sigma}\right)^{1/p} C_p.$$

Proof. We first consider the case $\sigma=\pi$. If $k(x)$ is such that $|x-k|\leqslant\frac{1}{2}$, then

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}} y_k \operatorname{sinc}\pi(x-k)\right| &\leqslant\left|\sum_{k\neq k(x)} y_k \operatorname{sinc}\pi(x-k)\right|+|y_{k(x)}\operatorname{sinc}\pi(x-k)|\\ &\leqslant\left|\sum_{k\neq k(x)} y_k\frac{1}{x-k}\right|+|y_k(x)|.\end{aligned}$$

Therefore it follows from Lemma 2.1 that we have

$$\|L_\pi y(x)\|_{p(\mathbb{R})}\leqslant\|Hy(x)\|_{p(\mathbb{R})}+\|y\|_{l_p}\leqslant C_p\,\|y\|_{l_p}. \tag{2.4}$$

By changing scale, we obtain from (2.4) that

$$\|L_\sigma y(x)\|_{p(\mathbb{R})}\leqslant\left(\frac{\pi}{\sigma}\right)^{1/p} C_p. \quad \blacksquare$$

LEMMA 2.3 [13, Theorem 6.7.1]. *Let* $g(z) \in E_\sigma$, $z = x + iy$, $g(x) \in B_{\sigma, p}$, $1 \leqslant p < \infty$. *Then*

$$\left(\int_{\mathbb{R}} |g(x+iy)|^p \, dx \right)^{1/p} \leqslant e^{\sigma |y|} \, \|g\|_{p(\mathbb{R})},$$

and if $|x| \to \infty$, *then* $g(x) \to 0$.

LEMMA 2.4 [13, Theorem 6.7.15]. *Let* $g \in B_{\sigma, p}$, $1 \leqslant p < \infty$. *Then*

$$\left(\frac{\pi}{\sigma} \sum_{k \in \mathbb{Z}} \left| g\left(\frac{k\pi}{\sigma} \right) \right|^p \right)^{1/p} \leqslant C_p \, \|g\|_{p(\mathbb{R})}, \qquad \sigma > 0.$$

LEMMA 2.5. *Let* $y = \{y_k\}_{k \in \mathbb{Z}}$, $y \in l_p$, $1 < p < \infty$. *Then the Whittaker series* $\sum_{k \in \mathbb{Z}} y_k \operatorname{sinc} \sigma(x - k\pi/\sigma)$ *is convergent uniformly on* $\mathbb{R}$. *If we make*

$$g(x) := \sum y_k \operatorname{sinc} \sigma(x - k\pi/\sigma) = L_\sigma y(x), \tag{2.5}$$

then $g(x) \in B_{\sigma, p}$ *and* $g(k\pi/\sigma) = y_k$, $k \in \mathbb{Z}$, *and*

$$|g(x)| \leqslant C_p \left\| \frac{\sin x}{x} \right\|_{q(\mathbb{R})} \|y\|_{l_p}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

$$\|g\|_{p(\mathbb{R})} \leqslant C_p \left(\frac{\pi}{\sigma} \right)^{1/p} \|y\|_{l_p}.$$

Proof. Following the method of [7, Lemma 3], we let $z = x + iy \in \mathbb{C}$ be fixed and let

$$h_\sigma(z, \eta) = \operatorname{sinc} \sigma(z - \eta), \qquad \eta \in \mathbb{C}, \quad \eta = \xi + i\zeta.$$

It follows from [6, p. 101] that as a function of η, $h_\sigma(z, \eta)$ is an entire function of exponential type σ. If $1 < p < \infty$, $1/p + 1/q = 1$, then $q > 1$ and we have

$$\left(\int_{\mathbb{R}} |h_\sigma(z, \eta)|^q \, d\zeta \right)^{1/q} = \left(\int_{\mathbb{R}} |\operatorname{sinc}(x + iy)|^q \, dx \right)^{1/q}$$

$$\leqslant \left(\frac{\pi}{\sigma} \right)^{1/q} e^{\sigma |y|} \left\| \frac{\sin x}{x} \right\|_{q(\mathbb{R})}, \tag{2.6}$$

therefore, by virtue of Hölder's inequality, (2.6), Lemma 2.3, and Lemma 2.4, we obtain

$$\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}} y_k h_\sigma(z, k\pi/\sigma)\right| \\
&\quad\leqslant \left(\sum_{k\in\mathbb{Z}} |h_\sigma(z, k\pi/\sigma)|^q\right)^{1/q} \|y\|_{l_p} \\
&\quad\leqslant C_q \left(\frac{\sigma}{\pi}\right)^{1/q} \|h_\sigma(z, k\pi/\sigma)\|_{q(\mathbb{R})} \|y\|_{l_p} \\
&\quad\leqslant C_q \left(\frac{\sigma}{\pi}\right)^{1/q} \cdot \left(\frac{\pi}{\sigma}\right)^{1/q} e^{\sigma|y|} \left\|\frac{\sin x}{x}\right\|_{q(\mathbb{R})} \|y\|_{l_p} \\
&\quad\leqslant C_q e^{\sigma|y|} \left\|\frac{\sin x}{x}\right\|_{q(\mathbb{R})} \|y\|_{l_p}. \qquad (2.7)
\end{aligned}$$

Let $g(z) := \sum_{k\in\mathbb{Z}} y_k h_\sigma(z, k\pi/\sigma)$. Equation (2.7) implies that the series $\sum_{k\in\mathbb{Z}} y_k h_\sigma(z, k\pi/\sigma)$ converges uniformly on all compact subsets of $\mathbb{C}$ and so its sum $g(z)$ defines an entire function and it follows from (2.7) that $g(z)\in E_\sigma$. Moreover, in view of Lemma 2.2, $g(x)\in L_p(\mathbb{R})$; therefore, $g(x)\in B_{\sigma,p}$ and

$$\|g(x)\|_p \leqslant C_p \left(\frac{\pi}{\sigma}\right)^{1/p} \|y\|_{l_p},$$

and it is clear that $g(k\pi/\sigma) = y_k$, $k\in\mathbb{Z}$. The proof of Lemma 2.5 is complete. ∎

Proof of Theorem 1. Let $f\in B_{\sigma,p}$ and let

$$g(x) := \sum_{k\in\mathbb{Z}} f(k\pi/\sigma)\,\mathrm{sinc}\,\sigma(x - k\pi/\sigma). \qquad (2.8)$$

By Lemma 2.4, the sequence $\{f(k\pi/\sigma)\}\in l_p$, hence in view of Lemma 2.5 the series on the right-hand side of (2.8) converges uniformly on $\mathbb{R}$ and $g\in B_{\sigma,p}$, $g(k\pi/\sigma) = f(k\pi/\sigma)$. Let $\delta(x) = f(x) - g(x)$ and let $\psi(z) = \delta((\pi/\sigma)\,z)$, $z = x + iy\in\mathbb{C}$. Then it is clear that $\psi(z)\in E_\pi$, $\psi(x)\in B_{\pi,p}$, $\psi(k) = 0$, $k\in\mathbb{Z}$; therefore, by a result of Pólya [13, Corollary 9.4.2], $\psi(z) = C_0 \sin \pi z$. In virtue of Lemma 2.3,

$$|\psi(x)| \to 0, \qquad |x| \to \infty,$$

hence $\psi(x) \equiv 0$, $f(x) = g(x)$ which together with (2.8) gives

$$f(x) = \sum_{k \in \mathbb{Z}} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma), \qquad \forall x \in \mathbb{R}.$$

Hence Part (a) of Theorem 1 holds. Now we prove Part (b) of Theorem 1. Let $f \in B_{\sigma, p}$, $1 < p < \infty$. By Lemma 2.4, $\{f(k\pi/\sigma)\}_{k \in \mathbb{Z}} \in l_p$, and it follows from Lemma 2.2 and Part (a) of Theorem 1 that

$$\begin{aligned} &\left\| f - \sum_{|k| \leqslant m} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma) \right\|_{p(\mathbb{R})} \\ &\quad = \left\| \sum_{|k| > m} f(k\pi/\sigma) \operatorname{sinc} \sigma(x - k\pi/\sigma) \right\|_{p(\mathbb{R})} \\ &\quad \leqslant C_p \left(\frac{\pi}{\sigma}\right)^{1/p} \left(\sum_{|k| > m} \left| f\left(\frac{k\pi}{\sigma}\right) \right|^p \right)^{1/p} \to 0, \qquad m \to \infty, \end{aligned}$$

which completes the proof of Part (b) of Theorem 1. From Part (a) and Lemma 2.4, we obtain Part (c) of Theorem 1. Thus Theorem 1 is proved. ▮

Proof of Theorem 2. Let $y = \{y_k\}_{k \in \mathbb{Z}} \in l_p$. In view of Lemma 2.5, there is a function $g(x) \in B_{\sigma, p}$ such that $g(k\pi/\sigma) = y_k$, $k \in \mathbb{Z}$. If there is a function $f \in B_{\sigma, p}$ such that $f(k\pi/\sigma) = y_k$, then in the same way as that for Theorem 1, we have $f(x) \equiv g(x)$, hence the first part of Theorem 2 is proved. On the other hand, if there is a $g \in B_{\sigma, p}$ such that $g(k\pi/\sigma) = y_k$, then from Lemma 2.4,

$$\|y\|_{l_p} = \left(\sum_{k \in \mathbb{Z}} \left| g\left(\frac{k\pi}{\sigma}\right) \right|^p \right)^{1/p} \leqslant C_p \left(\frac{\sigma}{\pi}\right)^{1/p} \|g\|_{p(\mathbb{R})} < +\infty.$$

Theorem 2 is proved. ▮

3. The Estimates for the Aliasing Error

Lemma 3.1. *Let* $f \in L_p(\mathbb{R})$, $\{f(k\pi/\sigma)\}_{k \in \mathbb{Z}} \in l_p$, $1 < p < \infty$. *Then for every* $g \in B_{\sigma, p}$, *we have*

$$\|f - L_\sigma(f)\|_{p(\mathbb{R})} \leqslant C_p \left(\frac{\pi}{\sigma} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) - g\left(\frac{k\pi}{\sigma}\right) \right|^p \right)^{1/p} + \|f - g\|_{p(\mathbb{R})}.$$

Proof. Let $g \in B_{\sigma,p}$. Using Theorem 1, $L_\sigma(g, x) \equiv g(x)$, so by Lemma 2.2 we have

$$\begin{aligned}\|f - L_\sigma(f)\|_{p(\mathbb{R})} &\leqslant \|L_\sigma(f) - L_\sigma(g)\|_{p(\mathbb{R})} + \|f-g\|_{p(\mathbb{R})}\\ &= \|L_\sigma(f-g)\|_{p(\mathbb{R})} + \|f-g\|_{p(\mathbb{R})}\\ &\leqslant C_p\left(\sum_{k\in\mathbb{Z}} \frac{\pi}{\sigma}\left|f\left(\frac{k\pi}{\sigma}\right) - g\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p} + \|f-g\|_{p(\mathbb{R})}\end{aligned}$$

which completes the proof Lemma 3.1. ▮

Proof of Theorem 3. From Lemma 3.1, we have the sufficiency of Theorem 3 immediately. The necessity of condition (a) is clear. Now we prove the necessity of condition (b) of Theorem 3. Assume that $\{g_\sigma\} \subset B_{\sigma,p}$ such that $\|f - g_\sigma\|_{p(\mathbb{R})} \to 0$, $\sigma \to +\infty$. If $\|f - L_\sigma(f)\|_{p(\mathbb{R})} \to 0$, then for a given $\varepsilon > 0$, there is a $\sigma_0 > 0$ such that, for all $\sigma \geqslant \sigma_0$,

$$\|f - L_\sigma(f)\|_{p(\mathbb{R})} \leqslant \frac{\varepsilon}{2}, \qquad \|f - g_\sigma\|_{p(\mathbb{R})} \leqslant \frac{\varepsilon}{2},$$

which together with Lemma 2.4 and Part (a) of Theorem 1 gives that if $\sigma \geqslant \sigma_0$, then

$$\begin{aligned}&\left(\frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k\pi}{\sigma}\right) - g\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p}\\ &\qquad\leqslant \|L_\sigma(f-g)\|_{p(\mathbb{R})} \leqslant \|L_\sigma(f) - f\|_{p(\mathbb{R})} + \|f-g\|_{p(\mathbb{R})} \leqslant \varepsilon.\end{aligned}$$ ▮

Let

$$K_r(t) = A_r\left(\frac{2r\sin(t/2r)}{t}\right)^{2r}, \qquad r \in \mathbb{N}, \quad A_r \in \mathbb{R}, \tag{3.1}$$

where the constant A_r is taken such that $\int_{\mathbb{R}} K_r(t)\,dt = 1$. It follows from [6, pp. 101–102] that $K_r(t) \in B_{1,1}$. Make

$$K_{r,\sigma}(t) = A_r\sigma\left(\frac{2r\sin(\sigma t/2r)}{\sigma t}\right)^{2r}, \tag{3.2}$$

then $K_{r,\sigma}(t) \in B_{\sigma,1}$ and

$$\int_{\mathbb{R}} K_{r,\sigma}(t)\,dt = 1. \tag{3.3}$$

LEMMA 3.2. *Let $h(t) \in L_p(\mathbb{R})$, $1 < p < \infty$, be a non-negative even function which is non-increasing on $[0, \infty)$. Let*

$$g(x) = \int_{\mathbb{R}} h(x+t)\, K_{2,\sigma}(t)\, dt, \qquad \sigma > 1.$$

Then there is a non-negative even function $\psi(x)$ which is non-increasing on $[0, \infty)$ such that

$$|g(x)| \leqslant C_{p,h}\psi(x), \qquad \forall x \in \mathbb{R},$$

where the constant $C_{p,h}$ depends on p and $h(x)$ only.

Proof. It is easy to prove that $g(x)$ is a non-negative and even function on $\mathbb{R}$. By [6, Theorem 3.6.2], $g \in B_{\sigma,p}$. Let $x > 1$ and

$$\begin{aligned} g(x) &= \left\{ \int_{-\infty}^{-2x} + \int_{-2x}^{-x/2} + \int_{-x/2}^{\infty} \right\} h(x+t)\, K_{2,\sigma}(t)\, dt \\ &:= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

If $t \in (-\infty, -2x)$, then $t < x + t < t/2 < 0$, and since $h(x)$ is non-negative and non-decreasing on $[-\infty, 0)$,

$$I_1(x) \leqslant \int_{-\infty}^{-2x} h\left(\frac{t}{2}\right) K_{2,\sigma}(t)\, dt \leqslant h(x) \int_{-\infty}^{-2x} K_{2,\sigma}(t)\, dt \leqslant h(x).$$

Let $1/p + 1/q = 1$. By Hölder's inequality and (3.2), if $\sigma > 1$, $x > 1$, we have

$$\begin{aligned} I_2(x) &\leqslant \left(\int_{-2x}^{-x/2} |h(x+t)|^p\, dt \right)^{1/p} \left(\int_{-2x}^{-x/2} |K_{2,\sigma}(t)|^q\, dt \right)^{1/q} \\ &\leqslant A_2\, \|h\|_{p(\mathbb{R})} \left(\int_{x/2}^{2x} \sigma^q \left| \frac{2 \sin \sigma t/2}{\sigma t} \right|^{4q} dt \right)^{1/q} \\ &\leqslant 2A_2\, \|h\|_{p(\mathbb{R})} \sigma^{1-1/q} \left(\int_{(1/4)\sigma x}^{\infty} \left(\frac{\sin t}{t} \right)^{4q} dt \right)^{1/q} \\ &\leqslant 2A_2\, \|h\|_{p(\mathbb{R})} \sigma^{1-1/q} \left(\int_{(1/4)\sigma x}^{\infty} \left(\frac{1}{t} \right)^{4q} dt \right)^{1/q} \\ &\leqslant 512 A_2\, \|h\|_{p(\mathbb{R})} \sigma^{-1} x^{-4+1/q} \\ &\leqslant 512 A_2\, \|h\|_{p(\mathbb{R})} x^{-4+1/q}. \end{aligned}$$

If $t \geqslant -\frac{1}{2}x$, then $t+x \geqslant x/2 > 0$. Since $h(x)$ is non-negative and non-increasing on $[0, \infty)$, $h(x+t) \leqslant h(x/2)$, and

$$I_3(x) \leqslant \int_{-x/2}^{\infty} h(x/2)\, K_{2,\sigma}(t)\, dt$$

$$\leqslant h(x/2) \int_{\mathbb{R}} K_{2,\sigma}(t)\, dt = h\left(\frac{x}{2}\right).$$

Let $C_{p,h} := \max\{512 A_2 \|h\|_{p(\mathbb{R})}, 2\}$, and $x > 1$. Then

$$0 \leqslant g(x) \leqslant h(x) + h\left(\frac{x}{2}\right) + 512 A_2 \|h\|_{p(\mathbb{R})} x^{-4+1/q}$$

$$\leqslant C_{p,h}\left(h\left(\frac{x}{2}\right) + x^{-4+1/q}\right).$$

On the other hand, from Hölder's inequality, we have

$$|g(x)| \leqslant \int_{\mathbb{R}} K_{1,\sigma}(t)\, dt \cdot \|h\|_{p(\mathbb{R})} = \|h\|_{p(\mathbb{R})}.$$

Let

$$\psi(x) = \begin{cases} \|h\|_{p(\mathbb{R})}, & \text{if } \ |x| \leqslant 1, \\ h\left(\dfrac{x}{2}\right) + |x|^{-4+1/q}, & \text{if } \ |x| > 1. \end{cases}$$

Then $\psi(x) \in L_p(\mathbb{R})$ and $\psi(x)$ is a non-negative even function which is non-increasing on $[0, \infty)$, and

$$|g(x)| \leqslant C_{p,h}\psi(x), \qquad \forall x \in \mathbb{R}. \quad \blacksquare$$

Let $\Delta_h^k f(t) = \sum_{j=0}^{k} C_k^j f(t+jh)$ be the kth difference, as a measure of the smoothness of the functions. We use the modulus of continuity with respect to the kth order difference, namely

$$\omega_k(f, t)_{p(\mathbb{R})} := \sup_{|h| \leqslant t} \|\Delta_h^k f(x)\|_{p(\mathbb{R})}. \tag{3.4}$$

LEMMA 3.3 [15, Chap. 5, 1.31]. *Let $f \in L_p(\mathbb{R})$, $1 \leqslant p < \infty$, $k \in \mathbb{N}$, $\sigma \geqslant 1$. Then there is an entire function $g_\sigma \in B_{\sigma,p}$ such that*

$$\|f - g_\sigma\|_{p(\mathbb{R})} \leqslant C_k \omega_k\left(f, \frac{1}{\sigma}\right)_{p(\mathbb{R})}.$$

Moreover, if $f \in L_p^1(\mathbb{R})$, *then*

$$\|f' - g'_\sigma\|_{p(\mathbb{R})} \leqslant C_k \omega_k \left(f', \frac{1}{\sigma}\right)_{p(\mathbb{R})}.$$

Proof of Theorem 4. Let $f \in \Omega_p$, $1 < p < \infty$. By the condition of the theorem, there is a non-negative function $h(x) \in L_p(\mathbb{R})$ which is non-increasing on $[0, \infty)$ such that $|f(x)| \leqslant C_0 h(x)$, $C_0 \in \mathbb{R}$. Let

$$g_\sigma(x) = \int_{\mathbb{R}} f(x+t)\, K_{2,\sigma}(t)\, dt,$$

where $K_{2,\sigma}(t)$ is defined by (3.3). Then $g_\sigma \in B_{\sigma,p}$, and

$$|g_\sigma(x)| \leqslant C_0 \int_{\mathbb{R}} h(x+t)\, K_{2,\sigma}(t)\, dt. \tag{3.5}$$

If $\sigma > 1$, then by Lemma 3.2 there is a non-negative even function $\psi(x) \in L_p(\mathbb{R})$ which is non-increasing on $[0, \infty)$ such that $|g_\sigma(x)| \leqslant C_{p,h} \psi(x)$. By Lemma 3.1, we obtain

$$\|f - L_\sigma(f)\|_{p(\mathbb{R})} \leqslant C_p \left(\frac{\pi}{\sigma} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) - g_\sigma\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p} + \|f - g_\sigma\|_{p(\mathbb{R})}. \tag{3.6}$$

For given $\varepsilon > 0$, since $h(x) \in L_p(\mathbb{R})$, $\psi(x) \in L_p(\mathbb{R})$, $1 < p < \infty$, there is a $M_0 > 0$, such that for all $M \geqslant M_0$,

$$C_0 C_{p,h} \left(\int_{|x| \geqslant M_0} |h(x)|^p\, dx\right)^{1/p} \leqslant \frac{\varepsilon}{4},$$

$$C_0 C_{p,h} \left(\int_{|x| \geqslant M_0} |\psi(x)|^p\, dx\right)^{1/p} \leqslant \frac{\varepsilon}{4}.$$

Let $\alpha(\sigma) = [\sigma M_0/\pi] + 1$ (here and hereafter $[a]$ denotes the integral part of $a \in \mathbb{R}$). Since $h(x)$ and $\psi(x)$ are even functions which are non-increasing on $[0, \infty)$, from the relation between series and integration, for all $\sigma > 0$, we have

$$\begin{aligned} C_{p,h} &\left(\frac{\pi}{\sigma} \sum_{|k| \geqslant \alpha(\sigma)} \left| f\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p} \\ &\leqslant C_0 C_{p,h} \left(\frac{\pi}{\sigma} \sum_{|k| \geqslant \alpha(\sigma)} \left| h\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p} \\ &\leqslant C_0 C_{p,h} \left(\int_{|x| \geqslant M_0} |h(x)|^p\, dx\right)^{1/p} \leqslant \frac{\varepsilon}{4}. \end{aligned} \tag{3.7}$$

By the same reason, we have

$$C_{p,h}\left(\frac{\pi}{\sigma}\sum_{|k|\geqslant \alpha(\sigma)}\left|g_\sigma\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p}\leqslant C_0\,C_{p,h}\left(\int_{|x|\geqslant M_0}|\psi(x)|^p\,dx\right)^{1/p}\leqslant\frac{\varepsilon}{4}. \quad (3.8)$$

On the other hand, since $f\in\mathfrak{R}$, f is Riemann integrable on $[-M_0, M_0]$, and it is clear that g_σ is Riemann integrable on $[-M_0, M_0]$. Therefore, there is a $\sigma_0>0$ such that for all $\sigma\geqslant\sigma_0$,

$$\begin{aligned}\left(\frac{\pi}{\sigma}\sum_{|k|\leqslant\beta(\sigma)}|f(k\pi/\sigma)-g_\sigma(k\pi/\sigma)|^p\right)^{1/p}&\leqslant\|f-g_\sigma\|_{p[-M_0,M_0]}+\frac{\varepsilon}{6}\\&\leqslant\|f-g_\sigma\|_{p(\mathbb{R})}+\frac{\varepsilon}{6},\end{aligned} \quad (3.9)$$

where $\beta(\sigma):=\alpha(\sigma)-1$. From (3.3) and (3.5), we obtain

$$f(x)-g_\sigma(x)=\int_{\mathbb{R}}(f(x)-f(t+x))\,K_{2,\sigma}(t)\,dt,$$

where $K_2(t)$ is defined by (3.1). Let

$$C^*:=\int_{\mathbb{R}}(1+|t|)\,K_{2,\sigma}(t)\,dt.$$

Then $C^*\in\mathbb{R}$, hence there is a $\sigma_1>1$ such that for all $\sigma>\sigma_1$ we have

$$\begin{aligned}\|f-g_\sigma\|_{p(\mathbb{R})}&\leqslant\omega\left(f,\frac{1}{\sigma}\right)_{p(\mathbb{R})}\int_{\mathbb{R}}(1+|t|/\sigma)\,K_2(t)\,dt\\&\leqslant\omega\left(f,\frac{1}{\sigma}\right)\int_{\mathbb{R}}(1+|t|)\,K_2(t)\,dt\\&\leqslant C^*\omega\left(f,\frac{1}{\sigma}\right)_{p(\mathbb{R})}\leqslant\frac{\varepsilon}{6},\end{aligned} \quad (3.10)$$

therefore if $\sigma\geqslant\max\{\sigma_0,\sigma_1\}$, from (3.6)–(3.10) we have

$$\|f-L_\sigma(f)\|_{p(\mathbb{R})}\leqslant\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\varepsilon. \quad ∎$$

LEMMA 3.4. *Let $f\in L_p^r(\mathbb{R})$, $r\in\mathbb{N}$, $1\leqslant p<\infty$. Then*

$$\left(\frac{\pi}{\sigma}\sum_{k\in\mathbb{Z}}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p}\leqslant\|f\|_{p(\mathbb{R})}+\frac{\pi}{\sigma}\|f'\|_{p(\mathbb{R})}<+\infty.$$

Proof. Let $x_k = k\pi/\sigma$, $k \in \mathbb{Z}$. By the mean value theorem, there is a $\xi_k \in [x_k, x_{k+1}]$, such that

$$|f(\xi_k)| = \frac{\sigma}{\pi} \int_{x_k}^{x_{k+1}} |f(u)|\, du,$$

therefore,

$$\begin{aligned} |f(x_k)| &\leqslant |f(\xi_k)| + |f(\xi_k) - f(x_k)| \\ &\leqslant \frac{\sigma}{\pi} \int_{x_k}^{x_{k+1}} |f(u)|\, du + \int_{x_k}^{x_{k+1}} |f(u)|\, du. \end{aligned} \tag{3.11}$$

By virtue of Stein's inequality [14], there is a constant C_0 which is independent of f such that

$$\|f'\|_{p(\mathbb{R})} \leqslant C_0 \, \|f\|_{p(\mathbb{R})}^{1-1/r} \, \|f^{(r)}\|_{p(\mathbb{R})}^{1/r} < +\infty. \tag{3.12}$$

Let $1/p + 1/q = 1$. From Hölder's inequality, (3.11), and (3.12), we get

$$\begin{aligned} \left(\frac{\pi}{\sigma} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) \right|^p \right)^{1/p} &\leqslant \left(\sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} |f(u)|^p \, du \right)^{1/p} \\ &\quad + \frac{\pi}{\sigma} \left(\sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} |f'(u)|^p \, du \right)^{1/p} \\ &\leqslant \|f\|_{p(\mathbb{R})} + \frac{\pi}{\sigma} \|f'\|_{p(\mathbb{R})} < +\infty. \quad \blacksquare \end{aligned}$$

Proof of Theorem 5. Let $f \in L_p^r(\mathbb{R})$, and let $g_\sigma(x)$ be defined by (3.5). From Lemma 3.1, Lemma 3.3, and Lemma 3.4, if $\sigma > 1$ we have

$$\begin{aligned} \|f - g\|_{p(\mathbb{R})} &\leqslant C_p \left(\frac{\pi}{\sigma} \sum_{k \in \mathbb{Z}} \left| f\left(\frac{k\pi}{\sigma}\right) - g_\sigma\left(\frac{k\pi}{\sigma}\right) \right|^p \right)^{1/p} + \|f - g_\sigma\|_{p(\mathbb{R})} \\ &\leqslant C_p \left(\|f - g_\sigma\|_{p(\mathbb{R})} + \frac{\pi}{\sigma} \|f' - g'_\sigma\|_{p(\mathbb{R})} \right) + \|f - g_\sigma\|_{p(\mathbb{R})} \\ &\leqslant C_{r,p} \omega_{r+1}\left(f, \frac{1}{\sigma}\right)_{p(\mathbb{R})} + \frac{\pi}{\sigma} \omega_{r+1}\left(f', \frac{1}{\sigma}\right)_{p(\mathbb{R})} \\ &\leqslant C_{r,p} \sigma^{-r} \left(\omega\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\mathbb{R})} + \omega_2\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\mathbb{R})} \right) \\ &\leqslant C_{r,p} \sigma^{-r} \omega\left(f^{(r)}, \frac{1}{\sigma}\right)_{p(\mathbb{R})}, \end{aligned}$$

which completes the proof of Theorem 5. $\blacksquare$

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